A Set of r Dynamical Attitude Equations for an Arbitrary n-Body Satellite Having r Rotational Degrees of Freedom

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A technique is developed for explicitly eliminating the constraint torques from a canonical system of n vector equations for the attitude dynamics of a satellite consisting of n arbitrarily interconnected rigid bodies. This elimination reduces the number of scalar second order differential equations from $3n$ to r, the number of degrees of rotational freedom of the satellite. At the same time, the number of dependent variables in these equations is reduced from the full set of $3n$ angular velocity components to just 3 such components for one body, together with $r - 3$ relative angular rates. This elimination and reduction saves computer time when the equations are integrated, and also avoids a possible build-up of numerical errors violating the constraints. The final equations resemble those obtained from a Lagrangian approach, but are simpler to derive and to modify to account for additional effects.

Introduction

A CANONICAL system of n vector equations for the attitude dynamics of a satellite consisting of n arbitrarily interconnected rigid bodies has been given by Hooker and Margulies.[1]† The only limitations on the configurations covered by their treatment were that chains of connected bodies should not form closed loops, and that only rotational motion was permitted at a joint (point of connection of two bodies). At any joint where this rotational motion has only one or two rotational degrees of freedom, there is in general a vector constraint torque orthogonal to the axes of rotation. Thus, if r is the total number of rotational degrees of freedom of the system, the number of nonzero constraint torque components for all the joints is in general $n_c = 3n - r$. Hooker and Margulies showed how to eliminate these constraint torque components in the sense that the full set of $3n$ differential equations for the angular velocity components was integrated in such a way that these velocities automatically satisfied the constraint equations. The calculation of the right hand sides of these differential equations required solving a system of $3n + n_c = 6n - r$ linear algebraic equations for the $3n$ angular velocity rates and the n_c constraint torques.

In contrast to the Eulerian approach used by Hooker and Margulies, the Lagrangian approach to deriving equations of motion has the advantage that constraint torques never appear, and that the number of dynamics differential equations is the same as r, the number of degrees of freedom. However, the Lagrangian approach suffers from the drawback that for a problem as complex as the general n-body satellite, it is exceedingly involved to derive the system kinetic and potential energies in a form suitable for subsequent differentiation in Lagrange's equations. Furthermore, in the Lagrangian formulation the equations are not written in terms of physical body axes, so it is sometimes unclear how to modify the equations—without rederiving them—to include such effects as those of additional internal forces and torques or of active control laws.

Thus it is desirable to find a set of r dynamical equations in which the constraint torques do not appear, and in which the physical body axes appear explicitly. This is accomplished in the present paper by means of a simple summation and projection method that allows the constraint torques to be explicitly eliminated from the Hooker-Margulies equations.‡ The essence of the method becomes clear if it is noted that the sum of the vector dynamical equations of all the bodies contains no constraint torque, since these action-and-reaction type torques all cancel in pairs. Thus we obtain 3 scalar equations by taking the components of this sum equation in some suitable frame. Next, we obtain one additional equation for each of the $r - 3$ gimbal axes as follows. Consider a typical gimbal axis, say at joint j. The vector constraint torque there can be isolated by summing the vector dynamical equations over all bodies that lie to one side of joint j. For then the constraint torques on these bodies all cancel in pairs, except for the one at joint j. But this remaining constraint torque must be orthogonal to the chosen gimbal axis (by the definition of a constraint). Thus the dot product of the summed equation with gimbal axis is a scalar equation which does not contain any constraint torque component.

The resulting differential system from the dynamics then consists of r scalar equations, and the evaluation of the right hand sides of these equations requires solving only r linear algebraic equations. (Whether the constraint torques are eliminated or not, there are at least r kinematical first-order differential equations to integrate; there will be more than r if redundant attitude coordinates such as Euler parameters or direction cosines are used.) The obvious advantage of eliminating the constraint torques is to reduce the computer time required for integrating the equations. Another advantage is the elimination of possible accumulation of numerical errors in the integration. These two advantages are borne out by the experience of Farrell, Newton, and Connelly,[6] whose simulation of the flexible booms of the Radio Astronomy Explorer (RAE) satellite did not explicitly eliminate the constraints. They attributed the slowness of their computer program in part to "the large number of integrated

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† Another derivation has been given by Roberson and Wittenburg in Ref. 2.

‡ A procedure for deriving the rotational equations of motion without constraint torques for any *specific* n-body configuration has been described by Russell in Ref. 3. However, Russell does not display the equations for a general configuration. Velman, in Ref. 4, gives a very brief sketch of a projection technique for deriving n-body equations of motion. His approach to eliminating constraint torques appears similar to the one used here; in addition, he implies that he uses the same technique to eliminate constraint forces. A general method for deriving equations of motion is described by Kane and Wang in Ref. 5. To my knowledge, it has not been used for the general n-body satellite.

variables in comparison with the number of independent coordinates."[6] Furthermore, to suppress possible growth of numerical errors caused by the constraint torque formulation, they introduced fictitious weak restraining springs and dampers on the locked joint axes.

The equations described in this paper have been programed for a digital computer, and were used for the configurations described in Ref. 7.

Vector Dynamical Equations with Constraint Torques

We consider a set of n rigid bodies connected in the form of a "topological tree"; that is, there are no closed loops. Thus there are $n - 1$ joints. At these joints we assume that only rotational motion is possible. Let the bodies be labelled uniquely (e.g., by integers), and let S be the set of their labels. Similarly, let the joints be labelled uniquely, and let J_λ be the set of labels for joints on body λ, for each $\lambda \epsilon S$. Further, for $\lambda \epsilon S$ and $j \epsilon J_\lambda$, let $S_{\lambda j}$ be the set of (labels of) those bodies connected to body λ at joint j, possibly via a chain of intermediate bodies.

For $\lambda \epsilon S$ and $j \epsilon J_\lambda$, let m_λ = mass of body λ, $m = \Sigma_{\nu \epsilon S} m_\nu$ = total mass of satellite, $\mathbf{\Phi}_\lambda$ = inertia dyadic of body λ about its center of mass, $\boldsymbol{\omega}_\lambda$ = angular velocity of body λ, $\boldsymbol{\mathfrak{L}}_{\lambda j}$ = vector from center of mass of body λ to joint j, $\boldsymbol{\rho}_\lambda$ = vector from Earth center to the center of mass of body λ, $\mathbf{F}_{\lambda j}$ = interaction force on body λ, transmitted through joint j, $\mathbf{T}_{\lambda j}{}^C$ = torque on body λ transmitted through joint j caused by gimbal constraints, $\mathbf{T}_{\lambda j}{}^{SD}$ = torque on body λ transmitted through joint j caused by springs, dampers, motors, etc., $\mathbf{F}_\lambda{}^E$ = external force on body λ (including gravity but not including $\mathbf{F}_{\lambda j}$), $\mathbf{T}_\lambda{}^E$ = external torque on body λ (including gravity gradient but not including $\mathbf{T}_{\lambda j}{}^C$ and $\mathbf{T}_{\lambda j}{}^{SD}$). A dot over a vector will denote its time derivative with respect to an inertial frame.

Newton's and Euler's equations for body λ are

$$\mathbf{F}_\lambda{}^E + \Sigma_{j\epsilon J_\lambda} \mathbf{F}_{\lambda j} = m_\lambda \ddot{\boldsymbol{\rho}}_\lambda$$

$$\mathbf{\Phi}_\lambda \cdot \dot{\boldsymbol{\omega}}_\lambda + \boldsymbol{\omega}_\lambda \times \mathbf{\Phi}_\lambda \cdot \boldsymbol{\omega}_\lambda = \mathbf{T}_\lambda{}^E + \Sigma_{j\epsilon J_\lambda}(\mathbf{T}_{\lambda j}{}^C + \mathbf{T}_{\lambda j}{}^{SD} + \boldsymbol{\mathfrak{L}}_{\lambda j} \times \mathbf{F}_{\lambda j})$$

respectively. Newton's equations can be used to eliminate the unknowns $\mathbf{F}_{\lambda j}$ from Euler's equations. As shown in Ref. 2, the result of the elimination can be written in the form

$$\Sigma_{\mu\epsilon S} \mathbf{\Phi}_{\lambda\mu} \cdot \dot{\boldsymbol{\omega}}_\mu = \mathbf{E}_\lambda + \Sigma_{j\epsilon J_\lambda} \mathbf{T}_{\lambda j}{}^C \text{ (for each } \lambda\epsilon S) \qquad (1)$$

in which

$$\mathbf{\Phi}_{\lambda\lambda} = \mathbf{\Phi}_\lambda + m_\lambda(D_\lambda{}^2\mathbf{1} - \mathbf{D}_\lambda\mathbf{D}_\lambda) + \Sigma_{j\epsilon J_\lambda} m_{\lambda j}(D_{\lambda j}{}^2\mathbf{1} - \mathbf{D}_{\lambda j}\mathbf{D}_{\lambda j})$$

$$\mathbf{\Phi}_{\lambda\mu}(\mu \neq \lambda) = -m[\mathbf{D}_{\mu p_\mu(\lambda)} \cdot \mathbf{D}_{\lambda p_\lambda(\mu)}\mathbf{1} - \mathbf{D}_{\mu p_\mu(\lambda)}\mathbf{D}_{\lambda p_\lambda(\mu)}]$$

$$\mathbf{E}_\lambda = 3\gamma\rho^{-3}\hat{\boldsymbol{\rho}} \times \mathbf{\Phi}_{\lambda\lambda} \cdot \hat{\boldsymbol{\rho}} - \boldsymbol{\omega}_\lambda \times \mathbf{\Phi}_{\lambda\lambda} \cdot \boldsymbol{\omega}_\lambda + \mathbf{T}_\lambda' + \Sigma_{j\epsilon J_\lambda}\mathbf{T}_{\lambda j}{}^{SD} + \mathbf{D}_\lambda \times \mathbf{F}_\lambda' + \Sigma_{j\epsilon J_\lambda}\mathbf{D}_{\lambda j} \times \Sigma_{\mu\epsilon S_{\lambda j}}\{\mathbf{F}_\mu' + m\boldsymbol{\omega}_\mu \times [\boldsymbol{\omega}_\mu \times \mathbf{D}_{\mu p_\mu}(\lambda)] + m\gamma\rho^{-3}(\mathbf{1} - 3\hat{\boldsymbol{\rho}}\hat{\boldsymbol{\rho}}) \cdot \mathbf{D}_{\mu p_\mu(\lambda)}\}$$

where, $m_{\lambda j} = \Sigma_{\mu\epsilon S_{\lambda j}} m_\mu$, $\mathbf{D}_\lambda = -m^{-1}\Sigma_{j\epsilon J_\lambda} m_{\lambda j}\boldsymbol{\mathfrak{L}}_{\lambda j}$, $\mathbf{D}_{\lambda j} = \mathbf{D}_\lambda + \boldsymbol{\mathfrak{L}}_{\lambda j}$, $\mathbf{1}$ = identity dyadic, $p_\mu(\lambda)$ = that joint of body μ which leads to body λ, $\mathbf{F}_\lambda'$ = nongravitational external force on body λ, $\mathbf{T}_\lambda'$ = nongravitational external torque on body λ, γ = Earth's gravitational constant, ρ = distance from Earth center to satellite's composite center of mass, $\hat{\boldsymbol{\rho}}$ = unit vector from Earth center toward satellite's composite center of mass. It should be noted that in the expression for $\mathbf{E}_\lambda$, the gravitational effects have been separated from the general external force and torque terms, leaving the nongravitational force $\mathbf{F}_\lambda'$ and torque $\mathbf{T}_\lambda'$. The gravitational terms in $\mathbf{E}_\lambda$ are those containing the factor $\gamma\rho^{-3}$.

The significance of the vectors $\mathbf{D}_\lambda$ and $\mathbf{D}_{\lambda j}$ and the dyadic $\mathbf{\Phi}_{\lambda\lambda}$ is as follows. If we augment the mass distribution of body λ by point masses $m_{\lambda j}$ (which equals the mass of all bodies attached to body λ via joint j) located at joints j, respectively, we obtain a new center of mass B_λ for the augmented body λ, and call it the *connection barycenter*. Then $\mathbf{D}_\lambda$ is the vector from B_λ to the old center of mass, $\mathbf{D}_{\lambda j}$ is the vector from B_λ to joint j, and $\mathbf{\Phi}_{\lambda\lambda}$ is the inertia dyadic of the augmented body λ about B_λ.

There is no correspondingly simple description of the dyadics $\mathbf{\Phi}_{\lambda\mu}$ for $\mu \neq \lambda$, since the vectors which define them are not fixed in the same bodies.

Since the $\mathbf{T}_{\lambda j}{}^C$ consist of unknown gimbal constraints (if there are fewer than 3 degrees of freedom at any joint), Eqs. (1) are not sufficient for determining the desired $\dot{\boldsymbol{\omega}}_\mu$'s. The approach of Hooker and Margulies in Ref. 1 was to augment Eq. (1) by constraint equations that expressed the fact that the constraint torque at a joint j is orthogonal to the difference of the angular velocity vectors of the two bodies adjacent to j. There is one such scalar constraint equation for each unknown constraint torque component. If there are n_c of these, then since Eq. (1) yields $3n$ scalar equations, there are in all $3n + n_c$ simultaneous linear equations to solve for the $3n$ $\dot{\omega}_\mu$ components and the n_c constraint torque components. (Usually these constraints are of no interest and would be discarded.) When the $\dot{\omega}_\mu$'s are integrated numerically to give ω_μ's, these do in theory satisfy the "built-in" constraint equations, but it is possible that small numerical violations of the constraint equations could accumulate as the integration proceeds. Although such an accumulation has never been observed to be unstable, the stability of this procedure has not been investigated. As mentioned earlier, Farrell, Newton, and Connelly[6] felt it worthwhile to suppress possible error accumulation by putting a fictitious weak spring and damper at each constraint axis.

In the next section a procedure will be described for eliminating the constraint torques from Eq. (1), so that only $r = 3n - n_c$ scalar dynamical equations need be solved simultaneously.

Explicit Elimination of the Constraint Torques

Select one body as the main body, and let its label be 0. Then $\boldsymbol{\omega}_0$ will be retained, while the other $\boldsymbol{\omega}$'s will be expressed in terms of ω_0 and relative rates.

The number of degrees of rotational freedom of the system relative to body 0 is $r - 3$. Let $\mathbf{g}_i$ ($i = 1,2,\ldots,r - 3$) be the axes (at the joints) about which rotation is possible. To relate the labels (i) of the $\mathbf{g}_i$ to those of the bodies, define

$$\varepsilon_{i\mu} = \begin{cases} 1 & \text{if } \mathbf{g}_i \text{ belongs to a joint anywhere on the chain of bodies connecting bodies } \mu \text{ and } 0 \\ 0 & \text{otherwise (e.g., if } \mu = 0). \end{cases}$$

With this notation we can write, for all $\mu\epsilon S$

$$\boldsymbol{\omega}_\mu = \boldsymbol{\omega}_0 + \textstyle\sum_{k=1}^{r-3} \varepsilon_{k\mu}\dot{\gamma}_k\mathbf{g}_k \qquad (2)$$

where γ_k is the angle of rotation about $\mathbf{g}_k$. In what follows, the range of summation for k will not be shown; it is always from 1 to $r - 3$.

By means of Eq. (2) we shall eliminate all the $\boldsymbol{\omega}_\mu$'s ($\mu \neq 0$) from the equations of motion, in favor of $\boldsymbol{\omega}_0$ and the $\dot{\gamma}_k$'s.

We begin by summing Eq. (1) over all the bodies $\lambda\epsilon S$, and substituting Eq. (2) (wherever the range of summation for λ and μ is not specified, it is understood to be all of S)

$$\Sigma_\lambda\Sigma_\mu\mathbf{\Phi}_{\lambda\mu} \cdot (\dot{\boldsymbol{\omega}}_0 + \Sigma_k\varepsilon_{k\mu}\ddot{\gamma}\mathbf{g}_k) = \Sigma_\lambda\mathbf{E}_\lambda{}^* \qquad (3)$$

where

$$\mathbf{E}_\lambda{}^* = \mathbf{E}_\lambda - \Sigma_\mu\mathbf{\Phi}_{\lambda\mu} \cdot \Sigma_k\varepsilon_{k\mu}\dot{\gamma}_k\dot{\mathbf{g}}_k \qquad (4)$$

We can rewrite Eq. (3) as

$$\mathbf{a}_{00} \cdot \dot{\boldsymbol{\omega}}_0 + \Sigma_k\mathbf{a}_{0k}\ddot{\gamma}_k = \Sigma_\lambda\mathbf{E}_\lambda{}^* \qquad (3a)$$

where $\mathbf{a}_{00}$ is the dyadic

$$\mathbf{a}_{00} = \Sigma_\lambda \Sigma_\mu \mathbf{\Phi}_{\lambda\mu} \tag{5}$$

and $\mathbf{a}_{0k}$ is the vector

$$\mathbf{a}_{0k} = \Sigma_\lambda \Sigma_\mu \varepsilon_{k\mu} \mathbf{\Phi}_{\lambda\mu} \cdot \mathbf{g}_k \tag{6}$$

Next, for fixed i, we sum Eq. (1) for all bodies λ that are connected beyond gimbal axis $\mathbf{g}_i$, relative to body 0; that is, over all bodies λ such that $\varepsilon_{i\lambda} = 1$. Thus the sum can be written $\Sigma_\lambda \varepsilon_{i\lambda}\{\quad\}$, over all $\lambda \epsilon S$. The $\mathbf{T}^C$ terms on the right all cancel in pairs, except for one corresponding to the joint of axis $\mathbf{g}_i$. This remaining $\mathbf{T}^C$ is of course orthogonal to the axis of rotation $\mathbf{g}_i$, so we have

$$\mathbf{g}_i \cdot \Sigma_\lambda \varepsilon_{i\lambda}\{\Sigma_\mu \mathbf{\Phi}_{\lambda\mu} \cdot (\dot{\boldsymbol{\omega}}_0 + \Sigma_k \varepsilon_{k\mu} \ddot{\gamma}_k \mathbf{g}_k) + \mathbf{E}_\lambda^*\} = 0 \tag{7}$$

which can be written

$$\mathbf{a}_{i0} \cdot \dot{\boldsymbol{\omega}}_0 + \Sigma_k a_{ik} \ddot{\gamma}_k = \mathbf{g}_i \cdot \Sigma_\lambda \varepsilon_{i\lambda} \mathbf{E}_\lambda^* \tag{7a}$$

and $\mathbf{a}_{i0}$ is the vector

$$\mathbf{a}_{i0} = \mathbf{g}_i \cdot \Sigma_\lambda \Sigma_\mu \varepsilon_{i\lambda} \mathbf{\Phi}_{\lambda\mu} \tag{8}$$

and a_{ik} is the scalar

$$a_{ik} = \mathbf{g}_i \cdot \Sigma_\lambda \Sigma_\mu \epsilon_{i\lambda} \epsilon_{k\mu} \mathbf{\Phi}_{\lambda\mu} \cdot \mathbf{g}_k \tag{9}$$

The dynamical Eqs. (3a) and (7a) can be written together as

$$\begin{bmatrix} \mathbf{a}_{00} & \mathbf{a}_{01} & \mathbf{a}_{02} & \cdots & \mathbf{a}_{0,r-3} \\ \mathbf{a}_{10} & a_{11} & \cdots & & \\ \mathbf{a}_{20} & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \mathbf{a}_{r-3,0} & \cdots & & & \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\omega}}_0 \\ \ddot{\gamma}_1 \\ \ddot{\gamma}_2 \\ \cdot \\ \cdot \\ \cdot \\ \ddot{\gamma}_{r-3} \end{bmatrix} = \begin{bmatrix} \Sigma_\lambda \mathbf{E}_\lambda^* \\ \mathbf{g}_1 \cdot \Sigma_\lambda \epsilon_{1\lambda} \mathbf{E}_\lambda^* \\ \mathbf{g}_2 \cdot \Sigma_\lambda \varepsilon_{2\lambda} \mathbf{E}_\lambda^* \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{g}_{r-3} \cdot \Sigma_\lambda \epsilon_{r-3,\lambda} \mathbf{E}_\lambda^* \end{bmatrix} \tag{10}$$

It is understood that where $\boldsymbol{\omega}_\mu$ occurs in $\mathbf{E}_\lambda^*$ on the right of Eq. (10), it is to be replaced according to Eq. (2).

When the vectors and dyadics in Eq. (10) are replaced by matrices of their components, the result is a set of r linear equations in the r variables: 3 components of $\dot{\boldsymbol{\omega}}_0$ and $\ddot{\gamma}_1$, $\ddot{\gamma}_2$, $\ldots$, $\ddot{\gamma}_{r-3}$. It is interesting that the $r \times r$ matrix on the left is symmetric; this follows from the fact that the component representation of $\mathbf{\Phi}_{\lambda\lambda}$ is symmetric and that the representations of $\mathbf{\Phi}_{\lambda\mu}$ and $\mathbf{\Phi}_{\mu\lambda}$ are transposes of each other.

Equation (10), together with the kinematic equations, constitute the complete equations of attitude motion. The kinematic equations consist of the trivial ones

$$d\gamma_i/dt = \dot{\gamma}_i \qquad (i = 1,2, \ldots, r-3)$$

and some equations for the main body such as the Euler angle rate equations or the direction cosine rate equations.

It is perhaps worthwhile to point out that if it is desired to compute a constraint torque (e.g. to monitor the stress on a joint), this can be done with quantities available in the present formulation. Suppose for example, that we desire the component $T_i{}^C = \mathbf{c}_i \cdot \mathbf{T}_{\nu j}{}^C$ where $\mathbf{c}_i$ is a direction at joint j about which relative motion is not permitted and ν is the body farther from body 0 at joint j. For notational consistency assume $i > r - 3$; then for this i, $\varepsilon_{i\lambda}$ can be defined as before, and we have, analogous to Eq. (7),

$$T_i{}^C = \mathbf{c}_i \cdot \Sigma_\lambda \varepsilon_{i\lambda}\{\Sigma_\mu \mathbf{\Phi}_{\lambda\mu} \cdot (\dot{\boldsymbol{\omega}}_0 + \Sigma_k \varepsilon_{k\mu} \ddot{\gamma}_k \mathbf{g}_k) - \mathbf{E}_\lambda^*\} \tag{11}$$

The $\dot{\boldsymbol{\omega}}_0$ and $\ddot{\gamma}$'s that occur here are available after Eq. (10) is solved.

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